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ORBIT-STABILIZER THEOREM AND CONSEQUENCES*

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Abstract

We present that a map $\backslash g$ from a nonempty set X into itself where each g is in a group G ; namely, is called an action of G on X. We next present the Orbit-Stabilizer Theorem: If G acts on X, $x \in X$ then the order of orbit of $x$is equal to the index of stabilizer of x in G . Last, we show that if an infinite group has a subgroup of finite index then it also has a normal infinite subgroup of finite index, which is the consequence of this theorem.

1. Basic Definitions and Examples

We begin by defining the group concept. Now for proper understanding of this paper we need to explain some terminologies used. Moreover, there are some basic definitions, examples, and Lagrange's Theorem (no proof), that use to back our points.

Definition 1.1.A **semigroup** is a non-empty set $X = \{\ldots, x, y, z, \ldots\}$ together with a binary operation $\circ$ which satisfies the following two conditions (axioms):

(i) it is **closed**, or **well-defined**: for all $x, y \in X$,

$$x \circ y \in X.$$

(ii) it is **associative**: for all $x, y, z \in X$,

$$(x \circ y) \circ z = x \circ (y \circ z).$$

For example, the set of all integers is a semigroup under usual addition.

Definition 1.2. A **group** $(G, \circ)$ is a semigroup, which satisfies the following extra conditions (axioms):

(iii) G contains a unique element e which satisfies, for all $g \in G$,

$$e \circ g = g \circ e = g$$

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(iv) for each $g \in G$, there exists a unique $g' \in G$ satisfying

$$g' \circ g = g \circ g' = e.$$

The element e is called the **neutral element** of the group G or the **identity** for G.

For example, the set of all integers is a group under usual addition with neutral element 0.

Definition 1.3. Two groups G and H are called **equal**, $G = H$, if and only if their underlying sets are equal and they have the same operation.

The **order** of a group G is the number (or cardinality) of elements in the underlying set of G, it is denoted by o(G).

For example, $G = \{-1, 1\}$ is a group under usual multiplication, with $o(G) = 2$.

Definition 1.4. If two numbers a and $b$have the property that their difference $a-b$ is integrally divisible by a number m(i.e., $a-b/m$ is an integer), then $a$and $b$are said to be **congruent modulo** m. The number m is called the **modulus**, and the statement "a is congruent to b (modulo m)" is written mathematically as

$$a \equiv b \,(\text{mod}\, m).$$

This notation was first introduced by C.F. Gauss in 1801 in his famous number theory text called 'Disquisitiones arithmeticae'. Next we define $a \sim b$ if $a \equiv b\,(\text{mod}\, m)$. Then this relation, ~, is an equivalence relation. Also define $[a] = \{b \in \mathbf{Z} \mid a \equiv b (\text{mod } m)\}$. Let $\mathbf{Z}_m$ denote the set $\{[0], [1], \dots, [m-1]\}$.

For example, $\mathbf{Z}_m \setminus \{[0]\}$ is a group under multiplicative modulo m where m is prime.

Definition 1.5. If X is a set and S_X denotes the collection of all permutations on X (that is, bijections of X *on*to itself), then this collection forms a group under the operation of composition, called the **symmetric group** on X. If X is finite with n elements, we usually take X to be the set $\{1, 2, \dots, n\}$ and write S_n for S_X, $o(S_n) = n!$.

For example, let $X = \{1, 2, 3\}$. Then we write all permutations from *X* onto itself in ray forms as follows:

$$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix},$$
$$\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix}.$$

We also write these ray forms in cycle notations as follows:

$$(1)(2)(3), \quad (23), \quad (12), \quad (123), \quad (132), \quad (13).$$

We see that (1)(2)(3) is the identity mapping; to simply, denote it by (1).(12), (13) and (23) are called **transpositions**. Note that every permutations in S_n is the product of transpositions. For instance, $(123) = (13)(12)$, and $(123)(456) = (13)(12)(46)(45)$.

If the number of transpositions of a permutation is **even** (**odd**), then it is called **even** (**odd**) permutation.

So $S_3 = \{(1), (12), (13), (23), (123), (132)\}$ is a group under the composition of permutations.

Definition 1.6. The symmetries of a regular polygon with *n* sides are the clockwise rotations about the centre are now by 2*π*/*n*, and 'reflection' or 'turning over' is as before. The set of all symmetries form a group under the composition of symmetries. This group is denoted by D_{2n}, called **dihedral**. We have $D_{2n} = \langle a, b : a^n = b^2 = e, b^{-1}ab = a^{-1} \rangle$.

For example, D_6 is the group of symmetries of the equilateral triangle, it has order 6, $D_6 = \langle a, b : a^3 = b^2 = e, b^{-1}ab = a^{-1} \rangle = \{e, a, a^2, b, ab, a^2b,\}$.

For example, the **alternating group** A_n which is the group contained in S_n of all even permutations; $A_3 = \{(1), (123), (132)\}$.

Most groups contain a number of smaller groups using the same operation, we shall, consider these now.

Definition 1.7. A **subgroup** H of a group G is a non-empty subset of G which forms a group under the operation of G.

We write $H \leq G$ when H is a subgroup of G. For example, if R^* is the group of all nonzero real numbers, then the group of all nonzero rational numbers, Q^*, is a subgroup of R^*, that is, $Q^* \leq R^*$.

Definition 1.8 (i) A subgroup J of a group G is called **proper** if $J \neq G$, this is denoted by $J < G$.

(ii) The singleton set $\{e\}$ forms a subgroup of all groups, it is called the **neutral subgroup** and is denoted by $\langle e \rangle$.

(iii) A subgroup H of a group G is called **maximal** in G if it is a**proper** subgroup of G, and whenever a subgroup J exists satisfying $H \leq J \leq G$, then either $J = H$ or $J = G$, so no subgroup lies strictly between H and G.

The neutral subgroup $\langle e \rangle$ is sometimes called the **identity**, **trivial**, or **unit**, subgroup.

(iv) A subset S of elements of a group G with the property that every element of G can be written as a finite product of elements of S and their inverses is called a set of **generators** of G, written by $G = \langle S \rangle$ and say G is **generated** by S or S **generates** G.

(v) A group G is called **cyclic** if there is an element a in G such that $G = \{a^n : n \in \mathbf{Z}\}$. Such an element a is called a **generator** of G, is written by $G = \langle a \rangle$.

Note that it is a subgroup of G.

(vi) If $H \leq G$ and $g \in G$, then $g^{-1}Hg$ is called a **conjugate subgroup** of H.

Note that $g^{-1}Hg \leq G$, for any $g^{-1}hg$, $g^{-1}h'g$ where h, h' are in H, we have

$$(g^{-1}hg)\,(g^{-1}h'g) = g^{-1}hh'g,$$

and

$$(g^{-1}hg)^{-1} = g^{-1}h^{-1}g;$$

hence $g^{-1}Hg \leq G$.

Definition 1.9. For $H \leq G$ and $g \in G$, the set $gH = \{gh : h \in H\}$ is called a **left coset** of H in G, and the set Hg is called a **right coset** of H in G.

Definition 1.10. Let $H \leq G$. The number (cardinality) of left cosets of H in G is called the **index** of H in G, it is denoted by $[G: H]$.

Note that this equals the number of right cosets of H in G.

Example1.11. Let $G = S_3 = \{(1),(12),(13),(23),(123),(132)\}$ and $H = \{(1),(12)\}$ Then $H < G$. The right cosets of $H$are

$$H(12) = H,$$
$$H(13) = \{(13),(132)\},$$
$$H(23) = \{(23),(123)\}$$

and so$[G : H] = 3$.

Definition 1.12. A group G is called **simple** if it contains no proper non-neutral normal subgroup.

For example, H in above example is simple.

Theorem 1.13(Lagrange's Theorem)If $H \leq G$ then $o(G) = o(H)[G : H]$.

Definition 1.14. If $H \leq G$, the **core** of H in G, core(H), is defined by

$$\text{core}(H) = \bigcap_{g \in G} g^{-1}Hg .$$

Proposition 1.15. Let G be a group and $H \leq G$. Then:

(i) core (H) is a normal subgroup of G.

(ii) core (H) is the largest normal subgroup of G contained in H.

Proof. (i) For any $x, y \in \text{core}(H)$, let $x = g^{-1}hg$ and $y = g^{-1}h_1g$ for all $g \in G, h, h_1 \in H$. Then

$$xy = (g^{-1}hg)(g^{-1}h_1 g) = g^{-1}hh_1 g$$

for all $g \in G$; so, $xy \in$core(H). Next, we have $x^{-1} = g^{-1}h^{-1}g$ for all $g \in G$, and so $x^{-1} \in$ core(H). Hence core (H)≤ G. Now we must show that core (H) satisfies normality. For any $g_1 \in G$,

$$g_1^{-1}xg_1 = g_1^{-1}g^{-1}hgg_1 = (gg_1)^{-1}hgg_1$$

for all $g \in G$; so, $g_1^{-1}xg_1 \in$ core(H).Hence core(H) $\triangleleft G$.

(ii) If N is any normal subgroup of G contained in H then we have $N = g^{-1}Ng \leq g^{-1}Hg$ for all $g \in G$ so that

$$N \leq \bigcap_{g \in G} g^{-1}Hg .$$

2. Actions

Given a set X, we introduce new collections of maps that transform X into itself and which are governed by a group G; that is, for each $g \in G$ we define a map $\backslash g : X \to X$, and map composition corresponds a group operation. The map $\backslash g$ is a permutation of X and it is called an **action** of G on X.

We begin by considering an example. Let G be the group $\mathbf{Z}_7 \setminus \{[0]\}$ and let $X = \{[1],[2[,[3],[4],[5],[6]\}$. Consider the right multiplication of an element x of X by an element g of G, that is, $x \cdot g$. Later we shall write this as $x \setminus g$. We have

$$x \setminus e = x,$$

that is, the right multiplication of element of X by e does not alter X. Also, by associativity we have

$$x \cdot (gh) = (x \cdot g) \cdot h,$$

where $g, h \in G$, that is, applying gh to x is the same as first applying g to x, and then applying h to the result. We call this procedure an action of G on X, see Definition 2.1 below. Further, as $[4] \in G$ we have

$[1]\cdot[4]=[4],\ [2]\cdot[4]=[1],\ [3]\cdot[4]=[5],\ [4]\cdot[4]=[2],\ [5]\cdot[4]=[6],\ [6]\cdot[4]=[3],$
that is, the set X has been permuted by this procedure. Also,

$[1]\cdot([4][3]) = [1]\cdot[5] = [5]$, and $([1]\cdot[4])\cdot[3] = [4]\cdot[3] = [5]$, gives $[1]\cdot([4][3]) = ([1]\cdot[4])\cdot[3]$.

This example above gives the following definition:

Definition 2.1. Given a nonempty set X and a group G, we say G **acts** on X if, for each $g \in G$, there exists a map $\backslash g : X \to X$, and these maps satisfy

$$\begin{aligned} &\text{(i)}\ x \backslash e = x \\ &\text{(ii)}\ x \backslash (gh) = (x \backslash g) \backslash h \end{aligned} \qquad (2.1)$$

for all $x \in X$ and $g, h \in G$. We call the map $\backslash g$ an instance of the **action** of the group G on the set X.

More formally, we can rewrite Definition 2.1, as follows: The function \ is a map from $X \times G$ to X which satisfies the two parts of (2.1), is called an **action** of G on X. The function defined above is a **right action**. Elements of group are denoted by the letters $g, h, \ldots$ and elements of set are denoted by x, y, z. We first prove the following basic result.

Theorem 2.2. Let G be a group and $g \in G$. Then the map $\backslash g : X \to X$ is a permutation of the set X.

Proof. Suppose $x, y \in X$ and $x \backslash g = y \backslash g$. Then by (2.1) we have

$$x = x \backslash e = x \backslash (gg^{-1}) = (x \backslash g) \backslash g^{-1} = (y \backslash g) \backslash g^{-1} = y \backslash (gg^{-1}) = y,$$

that is, the map $\backslash g$ is injective. Secondly, suppose $z \in X$ then for $g \in G$

$$z = z \backslash e = z \backslash (g^{-1}g) = (z \backslash g^{-1}) \backslash g,$$

that is, $z \backslash g^{-1}$ is preimage of z under the man $\backslash g$. Hence this map is also surjective, and so it is a permutation of X.

Examples 2.3. (a) Let G be a group and let X be the underlying set of G. The group G acts on X by right multiplication if we define, for $g \in G$ and $x \in X$,

$$x \backslash g = xg.$$

Then clearly satisfies the condition (2.1) and the corresponding action is called the **natural action** on G.

(b) Let V be a vector space defined over the field F and F^* the multiplicative group of F. For each $a \in \mathrm{F}^*$ define $\backslash a : V \to V$ by

$$v \backslash a = va.$$

Then the standard vector space axioms give

$$v \backslash 1 = v1 = v \text{ and } v \backslash (ab) = v(ab) = (va)b = (v \backslash a) \backslash b$$

where $b \in \mathrm{F}^*$; hence the multiplicative group F^* of F acts on V.

(c) Let $G = \langle e \rangle$ and X be an arbitrary set. Then G acts on X if we define $x \backslash e$ equal to x for all $x \in X$.

(d) Let $G = S_n$ and let $X = \{1, \dots, n\}$, then if we define $x \backslash \sigma = x\sigma$ for $\sigma \in G$ and $x \in X$. Then it is an action of G on X, called the **permutation action**.

There are two important entities concerning an action, namely, orbits and stabilizers. We first introduce orbits. Let the group G act on the set X. Define a relation $\sim$ on X by $x \sim y$ if $x \backslash g = y$ for some $g \in G$.

Lemma 2.4. The relation $\sim$ defined above is an equivalence relation.

Proof. Since $x \backslash e = x$, $x \sim x$. Suppose $x \sim y$, then $x \backslash g = y$ for some $g \in G$. We have

$$y \backslash g^{-1} = (x \backslash g) \backslash g^{-1} = x(gg^{-1}) = x \backslash e = x,$$

and so $y \sim x$. Finally, suppose $x \sim y$ and $y \sim z$ so that $x \backslash g = y$ and $y \backslash h = z$ for some $g, h \in G$. Then

$$x \backslash (gh) = (x \backslash g) \backslash h = y \backslash h = z,$$

that is, $x \sim z$. □

Definition 2.5. An equivalence class of the equivalence relation ~ given in Lemma 2.4 above is called an **orbit** of the action of G on X. The orbit containing the element $x \in X$ is called the **orbit** of x, and it is denoted by $\mathrm{O}_G\{x\}$.

An action of G on X is called **transitive** if there is only one orbit, that is, X itself, otherwise it is called **intransitive**.

We introduce the stabilizer, which is second new entity.

Definition 2.6. Given a group G acting on a set X, and $x \in X$, the subset of G,

$$\{g \in G : x \backslash g = x\},$$

is called the **stabilizer** of x in G; it is denoted by $stab_G(x)$.

The stabilizer of x is the elements of G whose associated maps do not move x.

Example 2.7. Let $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$ and $X = \{1, 2, 3\}$. Then

$$\begin{aligned} \mathrm{O}_G\{1\} &= \{y \in X : 1 \sim y\} \\ &= \{y \in X : 1 \backslash g = y, g \in G\} \\ &= \{1, 2, 3\}. \end{aligned}$$

Also, we have $\mathrm{O}_G\{2\} = \mathrm{O}_G\{3\} = X$. Next,

$$\begin{aligned} stab_G(1) &= \{g \in G : 1 \backslash g = 1\} \\ &= \{(1), (23)\}, \end{aligned}$$

$stab_G(2) = \{(1), (13)\}$ and $stab_G(3) = \{(1), (12)\}$.

Lemma 2.8. Let G be a group which acts on a set X. Then $stab_G(x) \le G$ for $x \in X$.

Proof. By definition of an action of G on X, $x \backslash e = x$, $e \in stab_G(x)$. If $x, y \in stab_G(x)$, then $x \backslash g = x$ and $x \backslash h = x$. Again, we have

$$x \backslash (gh^{-1}) = (x \backslash g) \backslash h^{-1} = (x \backslash h) \backslash h^{-1} = x,$$

so, $gh^{-1} \in stab_G(x)$.

By Example 2.7and Lagrange's theorem we have

$$o(O_G\{1\}) = o(G) / o(stab_G(1)) = [G : stab_G(1)],$$
$$o(O_G\{2\}) = o(G) / o(stab_G(2)) = [G : stab_G(2)],$$
$$o(O_G\{3\}) = o(G) / o(stab_G(3)) = [G : stab_G(3)].$$

This result forces to a general result, namely, the Orbit-Stabilizer Theorem.

Theorem 2.9. (Orbit-Stabilizer Theorem) If G acts on a set X, $x \in X$, and $O_G\{x\}$ is the orbit of x, then $o(O_G\{x\}) = [G : stab_G(x)]$.

Proof. By Lemma 2.8, $stab_G(x)$ is a subgroup of G. We define a map π from $O_G\{x\}$ to the set of right cosets of $stab_G(x)$ in G by

$$(x \backslash g)\pi = (stab_G(x))g,$$

where $g \in G$. We must show that this map is a bijection. First we show that this map is well-defined. Suppose $x \backslash g = x \backslash h$. Then

$$x \backslash (gh^{-1}) = (x \backslash g) \backslash h^{-1} = (x \backslash h) \backslash h^{-1} = x,$$

that is, $gh^{-1} \in stab_G(x)$. Hence $(stab_G(x))g = (stab_G(x))h$, as required. Second, we show that this map is injective. Suppose $(stab_G(x))g = (stab_G(x))h$, and so $gh^{-1} \in stab_G(x)$. Hence

$$x \backslash h = (x \backslash gh^{-1}) \backslash h = x \backslash (gh^{-1}h) = x \backslash g,$$

that is, π is injective. Last, we must show that π is surjective. For any $(stab_G(x))g$, $g \in G$ then there exists an element $x \backslash g \in O_G\{x\}$ such that

$$(x \backslash g)\pi = (stab_G(x))g,$$

that is, π is surjective. The theorem now follows.□

3. Applications

Now we calculate the order of the product of two subgroups of a group by using the Orbit-Stabilizer Theorem. Note that H and J are both non-normal subgroups of G, then HJ is not a subgroup of G, see the following example.

Example 3.1. Let $G = S_3$, $H = \{(1), (12)\}$ and $J = \{(1), (13)\}$. Then H and J are subgroups of G but $HJ = \{(1), (12), (13), (132)\}$ is not a subgroup of G, see the following table.

·	(1)	(12)	(13)	(132)
(1)	(1)	(12)	(13)	(132)
(12)	(12)	(1)	(132)	(13)
(13)	(13)	(123)	(1)	(23)
(132)	(132)	(23)	(12)	(123)

But if either H or J is a normal subgroup of G, HJ is a subgroup of G, see the following theorem.

Theorem 3.2. (a) If either H or J is a normal subgroup of G, then HJ is a subgroup of G and $HJ = JH$.

(b) If both H and J are normal subgroups of G, then HJ is a normal subgroup of G.

Proof.(a) Clearly HJ is a nonempty subset of G. Suppose $h_i \in H,\ j_i \in J,\ i = 1, 2$ and $H \lhd G$ (the proof is similar if $J \lhd G$). Since $H \lhd G$, $h_1^{-1}h_2 \in H$ and $j_1 \in J,\ j_1^{-1}(h_1^{-1}h_2)j_1 \in H.$ Let $j_1^{-1}(h_1^{-1}h_2)j_1 = h$ for some $h \in H$. Hence

$$(h_1 j_1)^{-1} h_2 j_2 = j_1^{-1} h_1^{-1} h_2 j_1 j_1^{-1} j_2 = h j_1^{-1} j_2 \in HJ,$$

that is, HJ is a subgroup of G. Now we show that $HJ = JH$. For any $h \in H$, $j \in J$, we have

$$hj = jj^{-1}hj .$$

$H \lhd G$ and above equation imply that $HJ = JH$.

(b) By (a) we only need to check normality. If $g \in G$, $h \in H$ and $j \in J$, we have $g^{-1}hjg = g^{-1}hg\, g^{-1}jg \in HJ$,

by hypothesis, the result follows.□

Theorem 3.3. If G is a finite group and H, J are subgroups of G, then

$$o(HJ)o(H \cap J) = o(H)o(J) .$$

Proof. We define an action. Let $X = \{Hg : g \in G\}$, the set of right cosets of H in G. The subgroup J acts on X by right multiplication if we set

$$Hg \setminus j = Hgj$$

for $j \in J$. This is an action since

$$Hg \setminus e = Hge = Hg$$

and

$$Hg \setminus j_1 j_2 = Hgj_1 j_2 = (Hg \setminus j_1) \setminus j_2$$

for $j_1, j_2 \in J$. The orbit of $O_J\{H\}$ of H is $\{H \setminus j : j \in J\}$, this equals HJ. Note that HJ is a disjoint union of right cosets of H, and the orbit of H under this action is the union of those cosets of H we can get to starting with H itself and applying elements $j \in J$. Hence

$$o(HJ) = o(H) \times o(O_J\{H\}).$$

Further, the stabilizer of H, $stab_J(H)$, equals $\{j \in J : Hj = H\}$, and so $stab_J(H) = H \cap J$. Hence, using the equation for $o(HJ)$ above, the Orbit-Stabilizer Theorem gives

$$\frac{o(HJ)}{o(H)} = o(O_J\{H\}) = [J : stab_J(H)] = \frac{o(J)}{o(H \cap J)} .$$

Example 3.4. Suppose $G = D_6 = \langle a,b : a^3 = b^2 = e, b^{-1}ab = a^{-1} \rangle$, $H = \langle b \rangle \le G$ and $J = \langle ab \rangle \le G.$ Then $o(H) = o(J) = 2,\ H \cap J = \{e\}$, and so $o(HJ) = 4$ by **Theorem 3.3**. Clearly, HJ is not a subgroup of G as $4 \nmid 6$. Also neither H nor J is normal, since $a^{-1}ba = ab$ is not in H and $a^{-1}aba = a^2b$, is not in J. But it is union of two cosets. For as

$$HJ = He \bigcup Hab = H \bigcup Hba^2 = H \bigcup Ha^2 .$$

Note that we also have $HJ = eJ \bigcup bJ = J \bigcup bJ$.

We have shown that (Theorem 2.2) that an action of a group G on a set X is a collection of permutations of X; that is, the action provides a map from G to S_X. Now we shall consider this map.

Definition 3.5. Let the group G act on the set X. The map $\nu : G \to S_X$ given by

$$g\nu = \backslash g$$

for all $g \in G$, is called the **permutation representation** of G for this action.

Lemma 3.6. The map ν given by Definition 3.5 is a homomorphism.

Proof. For all $g,\ h \in G$ and $x \in X$, we have by the definition of action of the group,

$$x \backslash (gh) = (x \backslash g) \backslash h = x(\backslash g \backslash h).$$

Again, by the definition of composition of functions, we have $\backslash gh = \backslash g \backslash h$

and hence

$$(gh)\nu = \backslash gh = \backslash g \backslash h = (g\nu)(h\nu),$$

as the required result.

This leads to the following result:

Theorem 3.7. Let G act on X with permutation representation ν as defined above. Then

$$\ker\nu = \bigcap_{x\in X} stab_G(x).$$

Proof. As ν is a homomorphism, its kernel is the set of those $g \in G$ for which $\backslash g$ is the identity permutation in S_X, that is, $x \backslash g = x$ for all $x \in X$. But $stab_G(x)$ is the set of those $g \in G$ for which $x \backslash g = x$, hence $\bigcap_{x\in X} stab_G(x)$ is the set of those $g \in G$ for which $x \backslash g = x$,

for all $x \in X$; that is, the kernel of ν .

Example3.8. Let $G = S_3$ in Example 2.7. Then $\bigcap_{x\in X} stab_G(x) = \langle(1)\rangle$. Hence the kernel of permutation representation in this case is $\langle(1)\rangle$, the neutral subgroup.

Referring again to Example 2.3(d), if $k \in \{1,2,\ldots,n\}$, then $stab_{S_n}(k)$ is the set of all permutation which fix k. Hence the intersection of these stabilizer for $k = 1, 2, \ldots, n$ is $\langle(1)\rangle$, and so the kernel of the permutation representation in this case is the neutral subgroup.

The converse of Lemma 3.6 is also valid as we show now.

Theorem 3.9. Suppose $\sigma : G \to S_X$ is a homomorphism of G to the group of all permutations on the set X. The map defined by $\backslash g = g\sigma$, for all $g \in G$. Then it is an action of G on X, and the permutation representation of this action is identical to σ.

Proof. Since σ is a homomorphism of G to the group of all permutations on the set X, $e\sigma$ is the identity permutation on X. So for all $x \in X$ and $g, h \in G$, we have

$$x \backslash e = x(e\sigma) = x$$

and

$$x \backslash (gh) = x((gh)\sigma) = x((g\sigma)(h\sigma)) = (x(g\sigma))(h\sigma) = (x \backslash g) \backslash h,$$

so that G acts on X. Now we show that the permutation representation of this action is identical to σ. Let ν be the permutation representation of this action, that is, for $g \in G$ and $x \in X$, $x \backslash g = x(g\nu)$. This gives $x(g\nu) = x(g\sigma)$ for all $x \in X$; hence $g\nu = g\sigma$ for all $g \in G$, which shows that $\nu = \sigma$. □

4. Restricted Actions

We find the relation of $stab_H(x)$ and $stab_G(x)$ when $H \le G$. Before finding it we first need to consider the subset of those elements which are fixed by an action.

Definition 4.1. Let G act on the set X. We set

$$fix(G,X) = \{x \in X : x \backslash g = x \text{ for all } g \in G\};$$

it is called the **fixed set** of X under the action of G.

Example 4.2. If G and X are given by the first example in Section 2, then $fix(G,X) = \varnothing$ but, if we change X to $X' = \{[1],[2],[3],[4],[5],[6],[7]\}$, then $fix(G,X) = \{[7]\}$. Also, $O_G\{[7]\} = \{[7]\}$ and $stab_G([7]) = G$.

Note that $fix(G,X)$ is a subset of X, and so it is not a group; for example, it is empty when the action is transitive. We have equivalent definitions

$$fix(G,X) = \{x \in X : O_G\{x\} = \{x\}\} = \{x \in X : stab_G(x) = G\}. \qquad (4.1)$$

Let G act on a set X and $H \le G$, we say H acts on a set X by restriction of the action of G on X. For example, if $G = \mathbf{Z}$, $H = 2\mathbf{Z}$, $X = G$ and the action of G on X is the natural one given by $x \backslash g = xg$, then the orbit of x under the action of G is the set of all integer multiples of x, whilst the orbit of x under the restricted action by H is the set of all even integer multiples of x.

We have, for $x \in X$ and $H \le G$,

$$stab_H(x) = stab_G(x) \cap H. \qquad (4.2)$$

In fact, if $g \in stab_H(x)$ then

$$x \backslash g = x \text{ and } g \in H,$$

which implies $g \in stab_G(x) \cap H$. Also, we have, if $g \in stab_G(x) \cap H$ then $x \backslash g = x$ and $g \in H$, which implies $g \in stab_H(x)$.

Lemma 4.3. If $H \leq G$, G acts on a set X and H acts on a set X by restriction of the action of G, then

$x \in fix(H, X)$ if and only if $H \leq stab_G(x)$.

Proof. We have

$x \in fix(H, X)$ if and only if $stab_H(x) = H$ by (4.1)

if and only if $stab_G(x) \cap H = H$ by (4.2)

if and only if $H \leq stab_G(x)$.

Lemma 4.4. Let $H \leq G$ and X the set of right cosets of H in G. Given $g \in G$ and $Hx \in X$, define

$$(Hx) \backslash g = Hxg. \tag{4.3}$$

Then it is a transitive action of G on X and $stab_G(Hx) = x^{-1}Hx$. Hence $[G : x^{-1}Hx] = [G : H]$.

Proof. We have, for $g, h \in G$ and $Hx \in X$,

$$Hx \backslash e = Hxe = Hx$$

and

$$((Hx) \backslash g) \backslash h = (Hxg) \backslash h = Hxgh = (Hx) \backslash gh;$$

hence it is an action. Further, it is a transitive action; for if $Hx, Hy \in X$, then

$$(Hx) \backslash x^{-1}y = (Hx)x^{-1}y = Hy;$$

so there is only one orbit, that is, X itself. Also

$$stab_G(Hx) = x^{-1}Hx$$

because

$$\begin{aligned} stab_G(Hx) &= \{g \in G : (Hx)\backslash g = Hx\} \\ &= \{g \in G : Hxg = Hx\} \\ &= \{g \in G : xgx^{-1} \in H\} \\ &= x^{-1}Hx. \end{aligned}$$

By using Orbit-Stabilizer Theorem, we have

$$[G : x^{-1}Hx] = [G : stab_G(Hx)] = o(X) = [G : H].$$

If ν_H is the permutation representation of this action, then by Theorem 3.7,

$$\ker \nu_H = \bigcap_{x \in G} x^{-1}Hx.$$

The entity on the right-hand side of this equation is called the **core** of H in G. If $[G : H] = n < \infty$, then $S_X \simeq S_n$, and ν_H gives a homomorphism from G into S_n. Hence we have

Theorem 4.5. (i) If $H<G$ and $[G : H] = n < \infty$, then there exists an injective homomorphism from G/core(H) = $G / \bigcap_{x \in G} x^{-1}Hx$ into S_n.

(ii) If $o(G/\text{ core}(H)) = m$, then $n \mid m$ and $m \mid n!$.

Proof. (i) Since $[G : H] = n < \infty$, G is the disjoint union of the collection of all right cosets of H in G. Let X be the set of right cosets of H in G. Given $g \in G$ and $Hx \in X$, define

$$(Hx)\backslash g = Hxg.$$

Then this is an action by Lemma 4.4. Next, we define $\nu_H : G \to S_n$ by

$$g\nu_H = \backslash g .$$

Then ν_H is a permutation representation of this action by Definition 3.5. So, by Lemma 3.6 ν_H is a homomorphism. Also, by Theorem 3.7, $\ker \nu_H = \bigcap_{x \in G} x^{-1}Hx$.

By definition of V_H it is onto, Hence by First Isomorphism Theorem, the proof is complete.

(ii) By (i) and assumption, $m = n!$, so that $m \mid n!$. By Proposition 1.15(ii), $H \leq \text{core}(H) \leq G$. We then have

$$\frac{|G|}{|H|} = \frac{|G|}{|core(H)|} \quad \frac{|core(H)|}{|H|}$$

$$n = m \quad \frac{|core(H)|}{|H|}$$

$$m = \frac{|H|}{|core(H)|} n,$$

which implies that $n \mid m$. □

Now we find the number of subgroups of a group by using above theorem.

Examples 4.6. (a) Now we find the number of subgroups of A_5. If G is simple and $H<G$, then

$$\bigcap_{x \in G} x^{-1} H x = \langle e \rangle,$$

as this intersection forms a normal subgroup of G contained in H. Hence by Theorem 4.5, there is an injective homomorphism from $G$to S_n, and so $o(G) \leq o(S_n) = n!$. Therefore, if $o(G) > n!$, G does not contain a subgroup (normal or not) of index n. For instance, consider $G = A_5$ with order 60. Suppose $H < A_5$, and $[A_5 : H] = n$. As $n! \geq 60$ only if $n > 4$, the theorem shows A_5 cannot have a subgroup of index 2, 3, or 4, so it cannot contain a subgroup of order 30, 20 or 15. In this case, we say that A_5 is not **reverse Lagrange**. It does contain a number of subgroups of order 12 (with index 5).

(b) The group A_4 is also not reverse Lagrange. For consider $G = A_4$ with order 12. Suppose $H < A_4$, and $[A_4 : H] = n$. As $n! \geq 12$ only if $n > 3$, the theorem shows A_4 cannot have a subgroup of index 2 or 3, so it cannot contain a subgroup of order 12 or 8. It does contain a number of subgroups of order 3 (with index 4).

We give an application of **Theorem 4.5** here.

Theorem 4.7. If G is an infinite group, $H \leq G$, and $[G : H] < \infty$, then G contains a normal subgroup K which satisfies $K \leq H$, and G/K is finite.

Proof. We have known that $g^{-1}Hg \leq G$. Clearly, $\bigcap_{x \in G} g^{-1}Hg \leq G$. Put $K = \bigcap_{x \in G} g^{-1}Hg$ in Theorem 4.5. Since this theorem gives an injective homomorphism of G into a finite symmetric group S_n, the factor group G/K is finite.□

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